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NOTE
SOLUTION OF AN EXTREMAL PROBLEM FOR SETS USING RESULTANTS OF POLYNOMIALS

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A new, short proof is given of the following theorem of Bollobás: Let $A_1, \dots, A_h$ and $B_1, \dots, B_h$ be collections of sets with $\forall i : |A_i| = r$, $|B_i| = s$ and $|A_i \cap B_j| = \emptyset$ if and only if $i = j$, then $h \le \binom{r+s}{s}$. The proof immediately extends to the generalizations of this theorem obtained by Frankl, Alon and others.

1. Introduction

In this paper we present a new and hopefully useful tool for proving results in extremal set theory. The result we present is not new, essentially it is Bollobás' theorem [3]. One of the most powerful generalizations is due to Alon [1], the relevance of the linear algebra bound in the context of Bollobás' Theorem was recognized by Lovász (1977B in Babai–Frankl [2]), earlier generalizations were obtained also by Kalai [4]. Alon's version reads as follows:

Theorem A. *Let $X_1, \dots, X_n$ be n sets, that are mutually disjoint and let $r_1, \dots, r_n$ and $s_1, \dots, s_n$ be positive integers. For $1 \le j \le h$ let A_j and B_j be subsets of $X = \cup X_i$ that satisfy*

$$\forall i, j : |A_i \cap X_j| \le r_i \qquad \text{and} \qquad |B_i \cap X_j| \le s_i,$$
$$A_i \cap B_i = \emptyset \qquad \text{and} \qquad A_i \cap B_j \ne \emptyset \qquad \text{for} \qquad i < j.$$

Then

$$h \le \prod_{i=1}^{n} \binom{r_i + s_i}{r_i}.$$

For clarity of exposition we start the proof with the case $n = 1$, and the more severe condition $A_i \cap B_j = \emptyset$ iff $i = j$. For further information, and interesting applications we refer to [1], and also chapter 5.2 of [2].

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2. Resultants of polynomials.

For two polynomials $\mathbf{a}(x)=a_0x^m+a_1x^{m-1}+\ldots+a_m$ and $\mathbf{b}(x)=b_0x^n+\ldots+b_n$ the *resultant* $\mathcal{R}(\mathbf{a},\mathbf{b})$ is by definition the value of the so–called Sylvester determinant

$$\mathcal{R}(\mathbf{a},\mathbf{b}) = \begin{vmatrix} a_0 & \cdots & a_m & & \\ & \ddots & & \ddots & \\ & & a_0 & \cdots & a_m \\ b_0 & \cdots & b_n & & \\ & \ddots & & \ddots & \\ & & b_0 & \cdots & b_n \end{vmatrix}$$

where there are n rows with $\mathbf{a}$'s and m rows with b's. We shall need the following properties of the resultant.

(1) $\mathcal{R}(\mathbf{a},\mathbf{b})$ is a homogeneous polynomial of degree n in the variables $a_0,\ldots,a_m$ and of degree m in $b_0,\ldots,b_n$.

(2) Unless both a_0 and b_0 are zero, $\mathcal{R}(\mathbf{a},\mathbf{b})=0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ have a common factor.

The first statement follows from the definition, the second can be found in most books on algebra, e.g. [5,p.444] or [6,p.103]. For completeness we give the alternative definitions of $\mathcal{R}(\mathbf{a},\mathbf{b})$ in the special case that a_0 and b_0 are nonzero: Let $\mathbf{a}(x) = a_0(x-\alpha_1)\ldots(x-\alpha_m)$ and $\mathbf{b}(x)=b_0(x-\beta_1)\ldots(x-\beta_n)$. Then

$$\mathcal{R}(\mathbf{a},\mathbf{b}) = a_0^n b_0^m \prod_{i,j}(\alpha_i-\beta_j) = b_0^m \prod_j a(\beta_j) = (-1)^{mn} a_0^n \prod_i b(\alpha_i).$$

3. Proof of the theorem of Bollobás.

Theorem [3]. *Let $A_1,\ldots,A_h$ and $B_1\ldots B_h$ be collections of sets with $\forall i: |A_i|=r$, $|B_i|=s$ and $A_i\cap B_j=\emptyset$ if and only if $i=j$. Then $h\le\binom{r+s}{s}$.*

Proof. Without loss of generality we have that all sets consist of real numbers. Define polynomials $\mathbf{a}_i(x)$, $i=1,\ldots,h$ of degree r by

$$\mathbf{a}_i(x) = \prod_{\alpha\in A_i}(x-\alpha).$$

In the same way define polynomials $\mathbf{b}_j(x)$, $j=1,\ldots,h$ of degree s. From the properties of the resultant and the conditions on the sets A_i and B_j we get that $\mathcal{R}(\mathbf{a}_i,\mathbf{b}_j)\neq 0$ if and only if $i=j$. Let $\mathbf{b}(x)$ be an arbitrary polynomial of degree s, then for a fixed i we consider $\mathcal{R}(\mathbf{a}_i,\mathbf{b})$ as a homogeneous polynomial of degree r in the coefficients $b_0,b_1,\ldots,b_s$ of $\mathbf{b}(x)$. The polynomials $\mathcal{R}(\mathbf{a}_i,\mathbf{b})$, $i=1,\ldots,h$ are independent in the $\binom{r+s}{s}$-dimensional vectorspace $\mathrm{Hom}(r;b_0,\ldots,b_s)$. To see this, suppose $\sum_i\lambda_i\mathcal{R}(\mathbf{a}_i,\mathbf{b})=0$. Then substituting for $\mathbf{b}$ the polynomial $\mathbf{b}_j$ we immediately get $\lambda_j=0$. From this the bound on h follows. ∎

4. Proof of the generalization due to Alon.

Proof of Theorem A. First observe that, by adding different new elements to our sets, we may assume $|A_i \cap X_j| = r_i$ etc. Define polynomials $\mathbf{a}_i(x)$ and $\mathbf{b}_i(x)$ as before and let

$$\mathcal{H} = \operatorname{Hom}(r_1, \dots, r_n; p_{10}, \dots, p_{1s_1}; p_{20}, \dots, p_{2s_2}; \dots; p_{n0}, \dots, p_{ns_n})$$

be the space of polynomials that are homogeneous of degree r_i in $p_{i0}, \dots, p_{is_i}$, $i = 1, \dots, n$. Note that

$$\dim \mathcal{H} = \prod_{i=1}^{n} \binom{r_i + s_i}{s_i}.$$

Associate with each set A_i as before a polynomial $\mathbf{a}_i(x)$, where we write

$$\mathbf{a}_i(x) = \prod_{j=1}^{n} \mathbf{a}_{ij}(x), \qquad \mathbf{a}_{ij}(x) = \prod_{\alpha \in A_i \cap X_j} (x - \alpha).$$

Define polynomials $\mathbf{b}_i(x) = \prod \mathbf{b}_{ij}(x)$ in the same way. Let $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$ be a sequence of polynomials, $\mathbf{p}_i(x) = p_{i0}x^{s_i} + p_{i1}x^{s_i - 1} + \dots + p_{is_i}$ of degree s_i and define for $1 \le i \le h$

$$\mathbf{f}_i := \prod_{j=1}^{n} \mathcal{R}(\mathbf{a}_{ij}, \mathbf{p}_j).$$

Clearly $\mathbf{f}_i \in \mathcal{H}$ and the polynomials $\mathbf{f}_i$ are independent: Suppose $\sum_i \lambda_i \mathbf{f}_i = 0$ and let k be maximal such that $\lambda_k \ne 0$. Substitute $\mathbf{p}_j = \mathbf{b}_{kj}$. Since $A_k \cap B_k = \emptyset$ and $A_j \cap B_k \ne \emptyset$ for $j < k$ we get that $\lambda_k = 0$, a contradiction. As a result we get the stated upper bound for h. ∎

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